Math Awareness Day – University of Houston – Victoria

Math Competition 2012

(Comments and questions: Dr. Ricardo Teixeira teixeirar@uhv.edu)

1. (7 points) Let *N* be the smallest positive integer such that when multiplied by 33 the result is a number where all digits are 7. What number is *N*?
(You may leave your answer as a fraction, without reducing it)

Answer: Since the resulting number is divisible by 3 and 11, we will use the divisibility rules for 3 and 11. The resulting number will only contain 7's, as 3 and 7 are relatively prime, then the quantity of 7's must be a multiple of 3 (in order to the sum of its digits be a multiple of 3). Also, the divisibility rule for 11 is that we have to add the odd-position digits and subtract the even-position digits, the result must be a multiple of 11. If the quantity of 7's is odd, then this rule would give "7", so the quantity of 7's must be even (and this rule will give "0", which is a multiple of 11). The smallest number then would be **777777/33**.

2. (5 points) A Texan farmer bought a rectangular shaped piece of land with measures 120m by 80m. Due to some environmental laws, he needs to plant trees on 20% of the land. He decides to do so by planting on two strips of equal width, according to the figure below. What is the width of each strip?

Answer: Total area is
$80m \, x \, 120m = 9600m^2$
Hence 20% of the area is
$0.20 \cdot 9600m^2 = 1920m^2$.
So, each strip is
$1920m^2 \div 2 = 960m^2$.
Each width is $960m^2 \div 120m = \mathbf{8m}$.

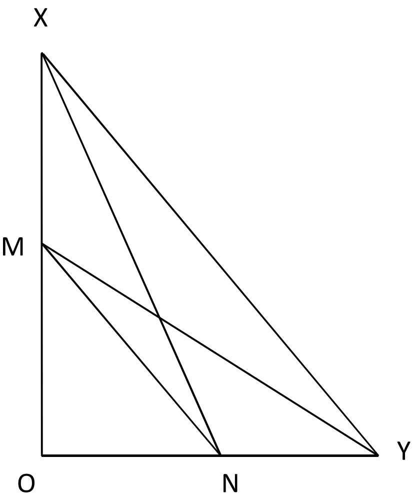
3. (8 points) How many non-congruent triangles with perimeter 7 will have all its sides with integer length?

Answer: Let a, b and c be the length of each side of the triangle, with $a \leq b \leq c \leq d$. Then $a + b + c = 7$. Possibilities are: $a = 1, b = 1, c = 5$; $a = 1, b = 2, c = 4$; $a = 1, b = 3, c = 3$; $a = 2, b = 2, c = 3$. Hence, $\mathbf{3}$ possibilities.

4. (5 points) Kathryn must make an 85 average on her tests to make an A in her math class. If she has 86, 91, 78, and 95 on her previous tests, what would the smallest grade that she need to make on her last test to get the A on her class?

Answer: Let x, be the grade she makes on her last exam. Her semester average has to be greater than 85, i.e.: $\frac{86+91+78+95+x}{5} \geq 85$. Then $\frac{350+x}{5} \geq 85 \Rightarrow 350 + x \geq 425 \Rightarrow x \geq \mathbf{75}$.

5. (10 points) Let XOY be a right triangle with $X\hat{O}Y = 90^o$. Let *M* and *N* be the midpoints of *OX* and *OY*, respectively. If $\overline{XN} = 19$ and $\overline{YM} = 22$, determine the length of the segment $\overline{MN}$.



Answer: Let $\overline{ON} = \overline{NY} = x$ and $\overline{OM} = \overline{MX} = y$. We are looking for the value of $\sqrt{x^2 + y^2}$. We know: $x^2 + (2y)^2 = 19^2$ and $(2x)^2 + y^2 = 22^2$. Then: $x^2 + 4y^2 = 361$ and $4x^2 + y^2 = 484$. Adding: $5x^2 + 5y^2 = 845$ so $x^2 + y^2 = 169$. Therefore:

$$\sqrt{x^2 + y^2} = \mathbf{13}.$$

6. (9 points) Mad at her boyfriend, Safira tore a love letter into n pieces. After that, she took one of the pieces and tore again into n smaller pieces. Since she was still mad, she got one of the last pieces and tore it into n even smaller pieces. From the numbers below, which could represent the final quantity of total pieces? 12, 23, 25, 30 or 38? (Hint: only one value is possible)

Answer: After the first "breakdown", Safira made n pieces. After the second she had $2n-1$ (since one piece became n). Finally, she ended up with $3n-2$ pieces. Hence, we need to find a number such that whenever we add 2, it becomes divisible by 3. The only number satisfying this property is $\mathbf{25}$.

7. (10 points) On a multiple choice test with 24 problems, each problem receives the following scores: four if the answer is correct, negative one if it is incorrect or zero if left blank. Robert received 52 points on the test, what is the maximum number of correct answers that he received.

Answer: Let c be the amount of correct answers, w be the amount of wrong answers and b be the amount of questions left blank. So, $c+w+b=24$. The amount of points is $4c-w=52$. So, $4c=52+w$. In other words, $52+w$ must be a number greater than or equal to 52 and divisible by 4. Since $w \leq 24$, the candidates for $4c$ are 52 (if $w=0$), 56 (if $w=4$), 60 (if $w=8$), 64 (if $w=12$), 68 (if $w=16$), 72 (if $w=20$) and 76 (if $w=24$). However, not every possibility is valid, since $c+w \leq 24$. Let's check: $w=0 \Rightarrow c=13$, $w=4 \Rightarrow c=14$, $w=8 \Rightarrow c=15$, as if $w \geq 9 \Rightarrow c>15$, we see that the maximum number of correct answers is $\mathbf{15}$.

8. (10 points) What is the third non-zero digit on the decimal representation of the fraction $\frac{1}{5^8}$? (Hint: There is a faster way other than making the computation)

Answer: The question becomes trivial is we make a small computation:

$\frac{1}{5^8} = \frac{1}{5^8} \cdot \frac{2^8}{2^8} = \frac{2^8}{10^8} = \frac{256}{10^8}$. Hence, the decimal representation of the number will have "6" as the third non-zero digit.

9. (9 points) Given the points A = (-7, 1), B = (2, 6), C = (7, 1) and D = (-2, -4). What term BEST describes the polygon ABCD?

a) Square;

b) Rectangle;

c) Rhombus;

d) Kite;

e) Parallelogram

f) none of previous

Answer: The slope of the line AB is $\frac{6-1}{2-(-7)} = \frac{5}{9}$. Slope of BC is $\frac{1-6}{7-2} = -\frac{5}{5} = -1$. Slope of CD is $\frac{-4-1}{-2-7} = \frac{-5}{-9} = \frac{5}{9}$. Slope of DA is $\frac{1-(-4)}{-7-(-2)} = \frac{5}{-5} = -1$. So, ABCD is a parallelogram. Also, since the intersecting lines are not perpendicular (otherwise the product of their slopes would be -1), then we know that ABCD is not a square or a rectangle. Because a "rhombus" is a parallelogram with all sides equal, we must verify if the adjacent sides have same length or not:

$$\overline{AB} = \sqrt{5^2 + 9^2}$$

$$\overline{BC} = \sqrt{(-5)^2 + 5^2}$$

Therefore, ABCD is just a **parallelogram**.

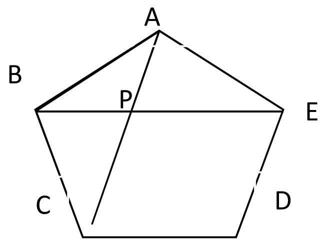
10. (6 points) How many different ways can you rearrange the letters in *school*?

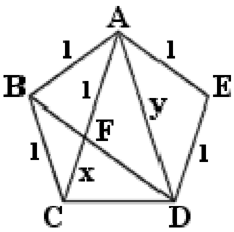
Answer: It is a combination/permutation problem. Since the order of the objects matters, it will be a permutation problem. If all the letters were different, the answer would be $6! = 6\cdot5\cdot4\cdot3\cdot2\cdot1 = 720$ permutations. Also, since there is a repetition (there are two letters "o"), changing "o" with "o" does not produce a new rearrangement. Hence we must divide by this number by $2! = 2\cdot1 = 2$. The answer is 360.

11. (10 points) Suppose there are two lines $\mathrm{y} = 6\mathrm{x}$ and $\mathrm{y} = -\frac{1}{6}\mathrm{x}$, and another line $\mathrm{y} = \mathrm{cx}$ with $\mathrm{c} > 0$ bisecting the angle between these two lines. Find the value of *c*.

Answer: Let A be the point (1,6), and B be (6,−1). Then A is on the first line, and B is on the second one. Also, if O is the origin (0,0), $\overline{OA} = \overline{OB} = \sqrt{1^2 + 6^2} = \sqrt{37}$. The triangle ΔAOB is then isosceles, and the angle $A\hat{O}B$ is the one we want to bisect. The line bisecting the angle will also bisect the side AB. The midpoint of AB is $\left(\frac{1+6}{2}, \frac{6+(-1)}{2}\right) = \left(\frac{7}{2}, \frac{5}{2}\right)$. Hence we need to find the equation of the line that passes through $(0,0)$ and $\left(\frac{7}{2}, \frac{5}{2}\right)$. It is $y = \frac{5}{7}x$. So, $c = \frac{5}{7}$.

12. (11 points) Suppose we have a regular pentagon with length of each side equal to 1. Find the length of $\overline{BP}$.





Answer: Each internal angle of the pentagon is 108 degrees. The diagonals split each angle of a regular pentagon into three identical angles. In this diagram, we see that quadrilateral AEDF is a rhombus (parallel sides and consecutive sides are congruent (equal)). And so, the diagonal AD bisects the angle EAF. And, by symmetry, all three angles are congruent (36 degrees).

Use the same diagram, Each side of our pentagon is 1. AF=1, because it is the side of our rhombus. Triangles ADF and BCF are similar. So, $\frac{x}{1} = \frac{1}{y}$. But $y = x + 1$. So $x = \frac{1}{x+1}$. Which gives $x(x+1) = 1$. So $x^2 + x - 1 = 0$. Solving for x, $x = \frac{-1 \pm \sqrt{5}}{2}$. Only one of these lengths is positive. So: $x = \frac{-1+\sqrt{5}}{2}$.